Chapter 10 Stability Analysis in Frequency Domain

§ 10.1 Stability in Frequency Domain

§ 10.2 Stability in Bode Plot

§ 10.3 Contour Mapping

§ 10.4 Nyquist Stability Analysis
§ 10.1 Stability in Frequency Domain (1)

- **Fundamental Behavior in Feedback:**

\[
G(j \omega) T(j \omega) = 1 + G(j \omega)H(j \omega)
\]

\[
\begin{align*}
T(j \omega) &= \frac{G(j \omega)}{1 + G(j \omega)H(j \omega)} \\
\text{If:} & \\
& \begin{cases} 
|G(j \omega)H(j \omega)| = 1 \\
\angle G(j \omega)H(j \omega) = -180^\circ
\end{cases}
\end{align*}
\]

- Sinusoidal Signal
- Phase Lag 180°
- Negative F.B.
§ 10.1 Stability in Frequency Domain (2)

- **Instability Condition:**
  \[ 1 + G(j\omega')H(j\omega') = 0 \]
  
  There exists a \( \omega' \) (critical frequency) to satisfy above equation.

  Stability conditions in magnitude and phase:
  
  \[ |G(j\omega')H(j\omega')| = 1 \quad ( \text{Gain} = 1 ) \]

  \[ \angle G(j\omega')H(j\omega') = -180^\circ \quad ( \text{Phase lag } 180^\circ ) \]

- **Absolute and Relative Stability:**

  Relative stability: Stability with margin in both gain and phase.
§ 10.1 Stability in Frequency Domain (3)

- **Relative Stability in Frequency Domain:**

1. Phase Margin

   \[ \text{P.M.} = \Phi + 180^\circ \]

   \( \Phi \) is obtained by the following procedure:

   1. Obtain \( \omega_g \) from \( |G(j\omega_g)H(j\omega_g)| = 1 \)

   \( \omega_g \) is called gain crossover frequency.

   2. \( \Phi = \angle G(j\omega_g)H(j\omega_g) \)

   The P.M. is that amount of “additional phase lag” as the crossover frequency \( (\omega_g) \) required to bring the system to the verge of stability.
§ 10.1 Stability in Frequency Domain (4)

2. Gain Margin

\[
G.M. = -20\log \left| \frac{G(j\omega_p)H(j\omega_p)}{} \right| \text{ (dB)}
\]

\[
= 20\log \left| \frac{1}{G(j\omega_p)H(j\omega_p)} \right| \text{ (dB)}
\]

\(\omega_p\) is obtained by:

\[
\angle G(j\omega_p)H(j\omega_p) = -180^\circ
\]

\(\omega_p\) is called phase crossover frequency

The G.M. is an increasing of the “tolerance gain degradation” due to the change of the input signal frequency when the phase lag is 180° (-180° phase shift).
§ 10.1 Stability in Frequency Domain (5)

- **Design Criterion:**

  Rule of thumb for stability margin:

  \[
  \begin{align*}
  \text{G.M.} & \geq 8\text{dB} & \text{or} & \text{G.M.} & \geq 6\text{dB} \\
  \text{P.M.} & \geq 30^\circ & & \text{P.M.} & \geq 40^\circ
  \end{align*}
  \]
§ 10.2 Stability in Bode Plot (1)

- **Stability in Bode Plot:**

\[
T(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}
\]

Critical conditions:

1. \(|G(j\omega)H(j\omega)| = 1\)

2. \(\angle G(j\omega)H(j\omega) = -180^\circ \ (180^\circ \text{ lag})\)
§ 10.2 Stability in Bode Plot (2)

- Stability Margin in Bode Plot:
§ 10.3 Contour Mapping (1)

- **Conformal Mapping:**

\[
F(s) = \frac{k \prod_{i=1}^{N} (s + s_i)}{\prod_{j=1}^{M} (s + s_j)}
\]

![Diagram showing contour mapping](image)
A contour map of a complex function $F(s)$ will only encircle the origin if the contour contains a pole or zero of the function.
**Cauchy Theorem (Principle of Argument):**

If a clockwise contour contains the number of poles, \( #P \), and zeros, \( #Z \), of a complex function, a contour mapping of the function will encircle the origin \( N \) times with \( N = #Z - #P \) (\( N > 0 \), clockwise, \( N < 0 \), counterclockwise).

Mapping by \( P(s) = F(s) - 1 \):

“Zero encirclement by \( F(s) \)” = “\(-1\) encirclement by \( P(s) \)”

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**Diagrams:**

- **F-plane** (\( F(s) \))
  - Contour \( \Gamma_F \)
  - Mapping by: \( 1+GH \)

- **P-plane** (\( P(s) \))
  - Contour \( \Gamma_p \)
  - Mapping by: \( GH \)

**Graphs:**

- Graph left shift “1”
- Graph right shift “1”
§ 10.3 Contour Mapping (4)

- **Contour Mapping by Vector Evaluation:**
  \[
  F(s) = 1 + G(s)H(s)
  \]
  \[
  P(s) = G(s)H(s) = \frac{500}{(s + 1)(s + 3)(s + 10)}
  \]

\[
V_1 \angle \phi_1 = 10.8 \angle 21.8^\circ
\]
\[
V_2 \angle \phi_2 = 5.0 \angle 53.1^\circ
\]
\[
V_3 \angle \phi_3 = 4.1 \angle 76.0^\circ
\]

\[
G(j\omega_b)H(j\omega_b) = \frac{500}{V_1 \angle \phi_1 \cdot V_2 \angle \phi_2 \cdot V_3 \angle \phi_3}
\]
\[
= 2.3 \angle -150.9^\circ
\]
§ 10.3 Contour Mapping (5)

\[
\begin{align*}
\omega &= 0, \ GH = \frac{50}{3}, \ \text{point} \ A' \\
\omega &\to \pm \infty, \ GH \to \frac{500j}{\omega^3}, \ \text{point} \ C' \ \text{and} \ D' \\
\omega &= \sqrt{43}, \ GH = -0.874 \ (\text{at Phase crossover frequency})
\end{align*}
\]
§ 10.4 Nyquist Stability Analysis (1)

- **Poles-zeros Between Open and Closed-loop Systems:**

\[
R(s) \quad + \quad G(s) \quad Y(s)
\]

\[
H(s) \quad -
\]

\[
T(s)
\]

\[
G(s) = \frac{N_G(s)}{D_G(s)} \\
H(s) = \frac{N_H(s)}{D_H(s)}
\]

\[
G(s)H(s) = \frac{N_G(s)N_H(s)}{D_G(s)D_H(s)}
\]

\[
1 + G(s)H(s) = \frac{D_G(s)D_H(s) + N_G(s)N_H(s)}{D_G(s)D_H(s)}
\]

\[
T(s) = \frac{N_G(s)D_H(s)}{D_G(s)D_H(s) + N_G(s)N_H(s)}
\]

“Poles of 1+GH” = “Poles of GH (Open-loop T.F.)”

“Zeros of 1+GH” = “Poles of T (Closed-loop T.F.)”
§ 10.4 Nyquist Stability Analysis (2)

- **Nyquist Method:**
  1. Nyquist plot
    \[ G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega), \omega = 0 \rightarrow \infty \]
  2. Construct Nyquist path (Include pole-zero diagram)

No poles on imaginary axis  A pole at origin  Poles on imaginary axis
§ 10.4 Nyquist Stability Analysis (3)

3. Mapping by $P(s) = G(s)H(s)$

   (1) Positive imaginary axis
   $0 < s < +j\omega$

   (2) Negative imaginary axis
   $-j\omega < s < 0$, $P(-j\omega) = -P(j\omega)$

   (3) Infinite semicircle in RHP
   $s = \text{Re}^{j\theta}$, $\lim_{R \to \infty} P(\text{Re}^{j\theta}) |_{\theta=90^\circ \sim -90^\circ}$

   (4) Infinitesimal semicircle detours poles on imaginary axis
   $s = r e^{j\theta}$, $\lim_{r \to 0} P(r e^{j\theta}) |_{\theta=-90^\circ \sim 90^\circ}$

4. Use Cauchy Theorem

   $#Z = N + #P$

   No. of closed-loop poles in RHP No. of “-1” encirclement No. of open-loop poles in RHP by $G(s)H(s)$
§ 10.4 Nyquist Stability Analysis (4)

Ex: \[ G(s) = \frac{0.2}{s(1+4s)(1+6s)} \]

Step 1: Define Nyquist path

Step 2: Mapping \( C_1, C_2, C_3, C_4 \)

- **C_1** mapping:
  \( G(j0) = -90 \)
  \( G(j\infty) = 0 \angle -270^\circ \)
  \( G(j\omega) = 0.2 \frac{(-10\omega^2) - (\omega^2 - 24\omega^3)}{100\omega^4 + (\omega^2 - 24\omega^3)^2} \)

- **C_2** mapping:
  \( s = \text{Re}^j\omega \)
  \( \lim_{R \to \infty} G(\text{Re}^j\omega) \bigg|_{\theta = 90^\circ - 90^\circ} = \text{Original point of GH-plane} \)

- **C_3** mapping:
  \[ G(-j\omega) = -G(j\omega) \]

- **C_4** mapping:
  \( s = r e^{j\theta}, \lim_{r \to 0} G(r e^{j0}) \bigg|_{\theta = 90^\circ - 90^\circ} = \infty \angle 90^\circ \to \angle 0^\circ \to \angle -90^\circ \)
5. Stability Criterion

(1) Open-loop unstable system (#P ≠ 0)

\[ \#Z = 0 \text{ iff } N = -\#P \]

A closed-loop system (T(s)) is stable:

The number of counterclockwise encirclements of the Nyquist plot about the -1 is equal to the number of unstable poles of open-loop transfer function (G(s)H(s)).

Note: The frequency at which the Nyquist diagram intersects -1 is the same frequency that the root locus cross the jω-axis.
§ 10.4 Nyquist Stability Analysis (6)

Ex: \( \frac{(s + 3)(s + 5)}{(s - 2)(s - 4)} \)

\[ \begin{align*}
R(s) & \rightarrow (s + 3)(s + 5) \\
E(s) & \rightarrow \frac{(s - 2)(s - 4)}{s + 3}(s + 5) \\
Y(s) & \rightarrow \frac{(s - 2)(s - 4)}{s + 3}(s + 5)
\end{align*} \]

\( \sigma \) - 5 - 3 2 4

\( j\omega \)

\( \omega = \sqrt{11} \)

\( \omega = 0 \)

\( -1.33 \)

\( -1 \)

\( \frac{15}{8} \)

\( 1 \)

\( \#P=2 \Rightarrow N=-2 \) for stable system
§ 10.4 Nyquist Stability Analysis (7)

(2) Open-loop stable system (#P=0)

#Z=0 iff N= 0

An open-loop stable system is stable if and only if the Nyquist plot does not encircle the \(-1\).

Ex: \( G(s)H(s) = \frac{1}{(s^2 + 2s + 2)(s + 2)} \)

For stable system, traversing the polar plot of \( G(j\omega)H(j\omega) \) in the direction of increasing \( \omega \), the \(-1\) point lies at the left of the point of closest approach.
§ 10.4 Nyquist Stability Analysis (8)

- **Stability Margin in Nyquist Plot:**

![Diagram of GH-plane, Nyquist diagram, and Unit Circle with annotations for G.M., P.M., Gain, and Lag degree.]

- Gain Margin (G.M.) = $20 \log a$
- Phase Margin (P.M.) = $\alpha$
- Lag "degree" = $(\text{Gain} - 1)$
- Frequency: $\omega \rightarrow \infty$
- Gain at $\omega = \frac{1}{a}$
- $20 \log \left( \frac{1}{a} \right)$ dB
- $0$ dB (Gain = 1)
- $-20 \log(a)$ (dB)
- $\omega \rightarrow 0$
- $\omega_p$, $\omega_g$
\section*{10.4 Nyquist Stability Analysis (9)}

- \textit{Unity Feedback System:}

\( \begin{align*}
R(s) &+ E(s) \quad \rightarrow \quad Y(s) \quad \text{through} \quad G(s) \\
\end{align*} \)

Error Response: \( \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \)

Output Response: \( \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \)

\( \omega \rightarrow \infty \)

\( \omega \rightarrow 0 \)

\( \omega \rightarrow \infty \)

\( -1 \)

Nyquist diagram

Unit Circle

GH-plane

\text{Nyquist Stability Analysis (9)}
§ 10.4 Nyquist Stability Analysis (10)

- **Closed-loop System with P Control:**

\[
R(s) + E(s) \rightarrow K \rightarrow G_P(s) \rightarrow Y(s)
\]

Error Response: \[
\frac{E(s)}{R(s)} = \frac{1}{1 + KG(s)}
\]

Output Response: \[
\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}
\]

Critical condition: \( K = 4 \)

Stable condition: \( 0 < K < 4 \)

When the open-loop transfer function multiplied by \( K \), the original Nyquist plot will be multiplied by \( K \) along radial direction.

![Nyquist plot diagram](image)